The International Innovations Journal of Applied Science (IIJAS) Vol. 2, No.1, 15-03-2025
المجلد الثاني العدد الاول 15-3-2025 (IIJAS) مجلة ابتكارات الدولية للعلوم التطبيقية
https://doi.org/10.61856/4cq33h32

SCIENTIFIC EVENTS GATE

The International Innovations Journal of Applied Science

Journal homepage: https://iijas.eventsgate.org/iijas

ISSN: 3009-1853 (Online)

New properties of a subclass for a multivalent meromorphic functions with an operator

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ARTICLE INFO

Article history:
Received 13 Dec. 2024
Revised 26 Dec. 2024,
Accepted 14 Jan. 2025,
Available online 15 Mar. 2025

Keywords:
multivalent meromorphic function
convolution properties
Rafid- operator
arithmetic mean
convex linear combinations

ABSTRACT

We note in this study that we have calculated the Rafid- operator on the multivalent meromorphic functions that belong to the class $\Delta\Gamma^{\Theta}(\rho)$ and which is as in $(\Delta 1)$. On the punctured unit desk $\mho^* = \{\varpi \in \mathbb{C}: 0 < |\varpi| < 1\}$. Introduced defined a Rafid operator by (Atshan et al., (2011)) and also (Rosy et al. (2013)) studied the same operator on the univalent meromorphic function. Now in this research we studied this operator on the multivalent meromorphic function and we obtain. if an operator $\mathfrak{I}_{\mathcal{M}}^{\Theta}: \Delta\Gamma^{\Theta}(\rho) \dashrightarrow \Delta\Gamma^{\Theta}(\rho)$ which is as in defined$\Delta 1$, and after entering the operator on the above functions in $(\Delta 1)$, we get a new functions $(\Delta 3)$. We also introduce a new subclass $\mathfrak{I}\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E}, q, d)$ for these functions with this operator and obtain the necessary and sufficient condition for functions to belong to this class. We also obtain new results for several properties of these functions in $\mathfrak{I}\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E}, q, d)$, including closed under arithmetic mean , closed under the combinations of convex linear and convolution properties for functions. These results are related to complex analysis in the theory of geometric functions.

1. Introduction

A Complex analysis, traditionally known as the theory of complex binary variables, is a branch of mathematical analysis that deals with binary numbers. It is useful in many areas of mathematics, including algebraic geometry, number theory, combinations, analytic geometry, and applied mathematics, as well as in yoga, encompassing disciplines such as hydrodynamics, thermodynamics, cumulative mechanics, and torsional geometry. Consequently, the use of complex analysis also has applications in engineering field such as aerospace engineering and mechanical engineering (Newton G. to Math. (2023)). Since a differentiable function of a complex variable is equal to the sum function given by its Taylor series (i.e., it is analytic), complex analysis is particularly concerned with the analytic function, that is, a function given locally by a convergent power series. There are real analytic functions and complex analytic functions. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not generally apply to real analytic functions.A meromorphic function on the punctured unit desk $\mho^* = \{\varpi \in \mathbb{C}: 0 < |\varpi| < 1\}$ of the complex plane is a solid function on all $\mathbb{C}$ except a set of isolated points. We studied the multivalent meromorphic function by calculating the Ravid factor on it, studied these functions with the operator in a new class, obtained the necessary and sufficient condition, studied some new properties and obtained new results for the functions of this class. We note previous studies of this function(Wang et al (2009), Panigrahi (2015)).

Previous research. (Atshan et al., (2011)) introduced the Rafid operator defined by Atshan and Buti and studied several properties of class functions. Various properties and functions have been studied by many researchers,

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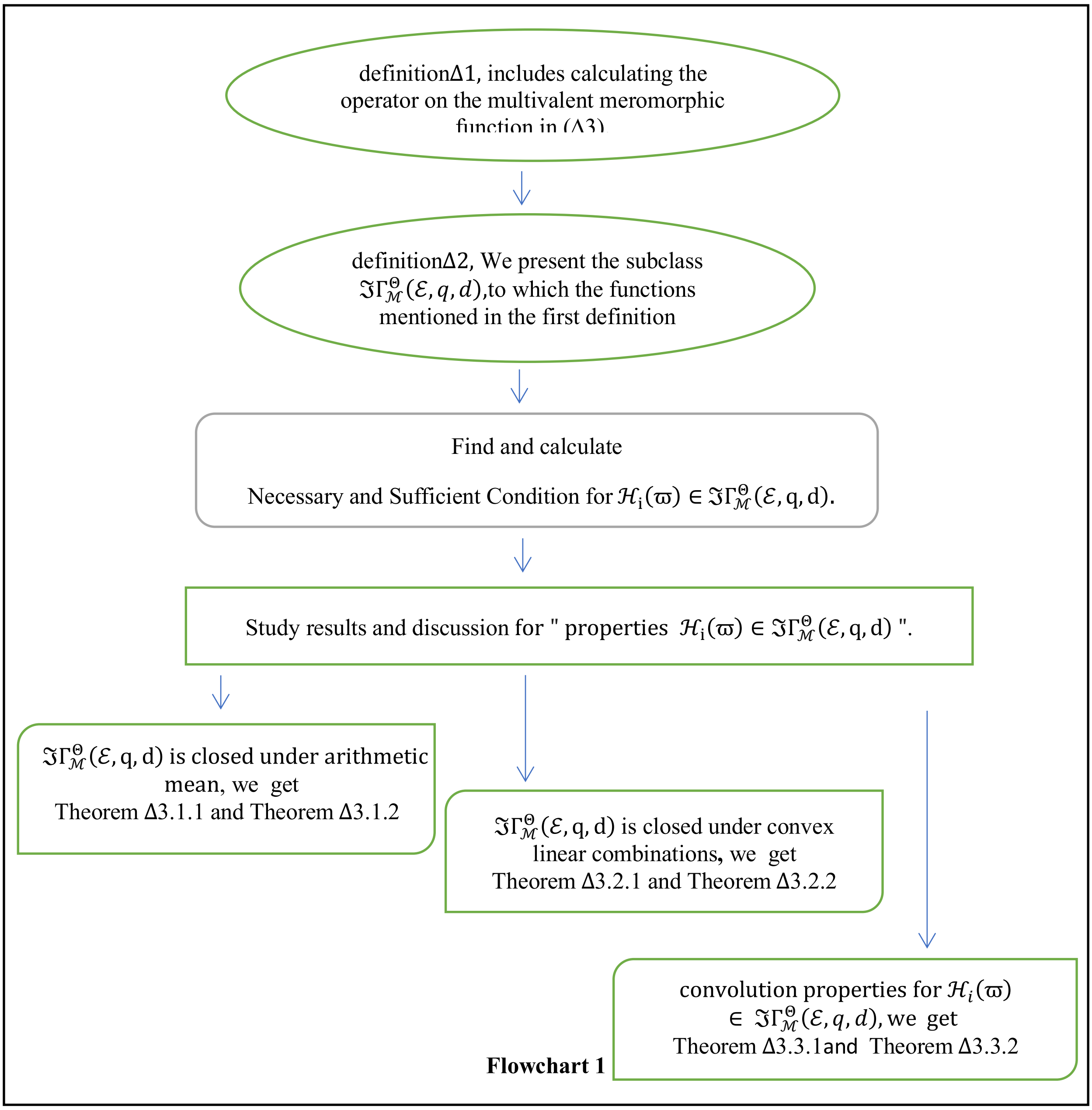
https://doi.org/10.61856/4cq33h32

including (Akgaul(, (Atshan et al. (2013), Atshan et al. (2013), Hussain S.K. et al. (2019), Liu et al. (2001) Mishra et al. (2014)) and they obtained different results and also (Rosy et al. (2013)) studied the same operator on the univalent meromorphic function. also studied many properties of this operator with the univalent meromorphic function.

meromorphic functions and also the factor that was studied previously and through these functions we will calculate the Rafid operator on them and obtain a new function under the influence of this factor and then we provide a definition of the new subcategory through which we obtain the affiliation of these functions to the category and the method of work is as follows in Flowchart 1.



Flowchart 1

Methodology. As we note (Rosy et al. (2013)) who calculated the Rafid operator but on monovalent functions, we collected many previous studies and took the multivalent

Now let $\Delta\Gamma^{\Theta}(\rho)$ is the class of meromorphic functions and which is as in:

$$\mathcal{H}_i(\varpi)=\frac{1}{\varpi^\rho}+\sum_{j=1}^{\infty}\mathrm{A}_{j+\rho,i}\,\varpi^{j+\rho},$$
$$i=1,2,\dots,\quad \rho\in\mathrm{N}=\{1,2,\dots\}. \qquad (\Delta1)$$

On the punctured unit desk $\mho^*=\{\varpi\in\mathbb{C}: 0<|\varpi|<1\}$.

Let $\mathcal{H}_1(\varpi)$ and $\mathcal{H}_2(\varpi)$ Provided by $(\Delta1)$ then $(\mathcal{H}_1*\mathcal{H}_2)(\varpi)$ denote of Hadamard product (or convolution), which is as in:

$$(\mathcal{H}_1*\mathcal{H}_2)(\varpi)=\frac{1}{\varpi^\rho}+\sum_{j=1}^{\infty}\mathrm{A}_{j+\rho,1}\mathrm{A}_{j+\rho,2}\,\varpi^{j+\rho}. \qquad (\Delta2)$$

DefinitionΔ1: $\forall\ \mathcal{H}_\mathrm{i}(\varpi)\in\Delta\Gamma^\Theta(\rho)$ in $(\Delta1)$, $0\le\mathcal{M}\le1$and $0\le\Theta\le1$, if an operator $\mathfrak{I}_\mathcal{M}^\Theta:\Delta\Gamma^\Theta(\rho)\dashrightarrow\Delta\Gamma^\Theta(\rho)$ which is as in:

$$\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i(\varpi)=\frac{1}{(1-\mathcal{M})^{\Theta-\rho}\Gamma(\Theta-\rho+1)}\int_0^\infty x^{\Theta-1}\,e^{-(\frac{x}{1-\mathcal{M}})}\mathcal{H}_i(\varpi x)dx,$$

then

$$\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i(\varpi)=\frac{1}{\varpi^\rho}+\sum_{j=1}^{\infty}\mathrm{A}_{j+\rho,i}\frac{(1-\mathcal{M})^{j+2\rho}\Gamma(j+\Theta+\rho+1)}{\Gamma(\Theta-\rho+1)}\,\varpi^{j+\rho}. \qquad (\Delta3)$$

And let

$$T(\Theta,\mathcal{M},\rho)=\frac{(1-\mathcal{M})^{j+2\rho}\Gamma(j+\Theta+\rho+1)}{\Gamma(\Theta-\rho+1)}.$$

DefinitionΔ2: $\forall\ \mathcal{H}_\mathrm{i}(\varpi)\in\Delta\Gamma^\Theta(\rho)$ in $(\Delta1)$, $i=1,2$, $0\le d<\rho$, $0\le q\le1$and $0<\mathcal{E}\le1$, let $\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},\mathrm{q},\mathrm{d})$ is subclass from $\Delta\Gamma^\Theta(\rho)$ then $\mathcal{H}_\mathrm{i}(\varpi)\in\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},q,d)$ if the condition is met:

$$\left|\frac{\rho+\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)}{(\rho-(1+q)d)-q\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)}\right|<\mathcal{E}. \qquad (\Delta4)$$

Where $0\le\mathcal{M}\le1$and $0\le\Theta\le1$.

2. "Necessary and Sufficient Condition for $\mathcal{H}_\mathrm{i}(\varpi)\in\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},q,d)$".

In this section, the necessary condition for the functions (Δ1)to belong to the class $\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},q,d)$ is calculated.

TheoremΔ2.1. $\forall\ \mathcal{H}_\mathrm{i}(\varpi)\in\Delta\Gamma^\Theta(\rho)$ in $(\Delta1)$, $i=1,2$, then $\mathcal{H}_\mathrm{i}(\varpi)\in\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},q,d)$ if and only if

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,i}\le\mathcal{E}(1+q)(\rho-d), \qquad (\Delta5)$$

where $0\le\mathcal{M}\le1$and $0\le\Theta\le1$.

A sharp result of the functions

$$\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i(\varpi)=\frac{1}{\varpi^\rho}+\sum_{j=1}^{\infty}\frac{\mathcal{E}(1+q)(\rho-d)}{(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}\,\varpi^{j+\rho}. \qquad (\Delta6)$$

Proof Δ2.1. Let (Δ5) be true, $|z|=1$

$$\left|\rho+\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)\right|$$
$$<\mathcal{E}\left|(\rho-(1+q)d)-q\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)\right|$$
$$=\left|\rho+\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)\right|$$
$$-\mathcal{E}\left|(\rho-(1+q)d)-q\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)\right|$$
$$=\left|\sum_{j=1}^{\infty}\mathrm{A}_{j+\rho,i}(j+\rho)T(\Theta,\mathcal{M},\rho)\,\varpi^{j+\rho}\right|$$
$$-\mathcal{E}\left|(\rho-(1+q)d)+q\rho-q\sum_{j=1}^{\infty}\mathrm{A}_{j+\rho,i}(j+\rho)T(\Theta,\mathcal{M},\rho)\,\varpi^{j+\rho}\right|$$
$$\le\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,i}-\mathcal{E}(1+q)(\rho-d)<0.$$

From (Δ5), we get $\mathcal{H}_\mathrm{i}(\varpi)\in\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},q,d)$.
On the contrary, let us $\mathcal{H}_\mathrm{i}(\varpi)\in\mathfrak{I}\Gamma_\mathcal{M}^\Theta(\mathcal{E},q,d)$ and from (Δ4), we have

$$\left|\frac{\rho+\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)}{(\rho-(1+q)d)-q\varpi^{\rho+1}\,\mathfrak{I}_\mathcal{M}^\Theta\mathcal{H}_i'(\varpi)}\right|<\mathcal{E},$$

this means that

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https://doi.org/10.61856/4cq33h32

$$\left|\frac{\sum_{j=1}^{\infty} A_{j+\rho,i}(j+\rho)T(\Theta,\mathcal{M},\rho)\,\varpi^{j+\rho}}{(\rho-(1+q)d)+q\rho-q\sum_{j=1}^{\infty} A_{j+\rho,i}(j+\rho)T(\Theta,\mathcal{M},\rho)\,\varpi^{j+\rho}}\right| < \mathcal{E}.$$

And because $Re(\varpi) < |\varpi|$, we get

$$Re\left\{\frac{\sum_{j=1}^{\infty} A_{j+\rho,i}(j+\rho)T(\Theta,\mathcal{M},\rho)\,\varpi^{j+\rho}}{(\rho-(1+q)d)+q\rho-q\sum_{j=1}^{\infty} A_{j+\rho,i}(j+\rho)T(\Theta,\mathcal{M},\rho)\,\varpi^{j+\rho}}\right\} < \mathcal{E}.$$

From $\varpi \to 1^-$ throughout real values. Then we get (Δ5)

Corollary Δ2.2 : $\forall\ \mathcal{H}_i(\varpi) \in \Delta\Gamma^{\Theta}(\rho)$ in (Δ1), $i = 1,2$, if $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$, then

$$A_{j+\rho,i} \le \frac{\mathcal{E}(1+q)(\rho-d)}{(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

A sharp result of the functions in (Δ6).

3. Results and discussion for " properties $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ ".

Now we study the properties of the class $\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ and we get the results that the class is closed under the arithmetic mean, closed under the combinations of convex linear and the properties of the convolution of functions.

Δ3. 1. " $\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ is closed under arithmetic mean ".

Theorem Δ3.1.1. Let it be the functions $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall\ i = 1,2,\dots,r$. Then the function $\mathcal{HH}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$,

$$\mathcal{HH}_i(\varpi) = \frac{1}{\varpi^{\rho}} + \sum_{j=1}^{\infty} AA_{j+\rho}\,\varpi^{j+\rho}, \qquad i = 1,2,\dots,r, \qquad (\Delta7)$$

where

$$AA_{j+\rho} = \frac{1}{\gamma}\sum_{i=1}^{\gamma} A_{j+\rho,i}\ , \qquad j = 1,2,\dots .$$

proof Δ3.1.1. From $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ and from Theorem Δ2.1, we get

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)A_{j+\rho,i} \le \mathcal{E}(1+q)(\rho-d),$$

$\forall\ i = 1,2,\dots,r$. So

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)AA_{j+\rho,i}$$
$$= \sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\left(\frac{1}{\gamma}\sum_{i=1}^{\gamma} A_{j+\rho,i}\right)$$
$$= \frac{1}{\gamma}\sum_{i=1}^{\gamma}\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)A_{j+\rho,i}$$
$$\le \frac{1}{\gamma}\sum_{i=1}^{\gamma}\mathcal{E}(1+q)(\rho-d) = \mathcal{E}(1+q)(\rho-d),$$

we get $\mathcal{HH}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$.

Theorem Δ3.1.2. Let it be the functions $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E}_i,q_i,d_i)$ in(Δ1), $\forall\ i = ,2,\dots,r$, and $0 \le d < \rho$, $0 \le q \le 1$and $0 < \mathcal{E} \le 1$ Then the function $\mathcal{HH}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in (Δ7). Where $\mathcal{E} = min\{\mathcal{E}_i\}, q = min\{q_i\}, d = min\{d_i\}$ and $1 \le i \le \gamma$.

proof Δ3.1.2. From $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E}_i,q_i,d_i)$in ($\Delta$1), $\forall\ i = 1,2,\dots,r$, and from Theorem Δ2.1, we get

$$\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)A_{j+\rho,i} \le \mathcal{E}_i(1+q_i)(\rho-d_i),$$

so

$$\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\left(\frac{1}{\gamma}\sum_{i=1}^{\gamma} A_{j+\rho,i}\right)$$
$$= \frac{1}{\gamma}\sum_{i=1}^{\gamma}\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)A_{j+\rho,i}$$
$$\le \frac{1}{\gamma}\sum_{i=1}^{\gamma}\mathcal{E}_i(1+q_i)(\rho-d_i),$$
$$\le \mathcal{E}(1+q)(\rho-d).$$

Hence $\mathcal{HH}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$.

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https://doi.org/10.61856/4cq33h32

Table 1: Comparison between Theorem Δ3.1.1 **and Theorem** Δ3.1.2
($\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ is closed under arithmetic mean)

theorem	let	where	then
Δ3.1.1	$\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall\, i = 1,2,\dots,r$.	$\mathcal{HH}_i(\varpi) = \frac{1}{\varpi^\rho} + \sum_{j=1}^{\infty} \mathrm{AA}_{j+\rho}\, \varpi^{j+\rho}$, $i = 1,2,\dots,r$, (Δ7) where $\mathrm{AA}_{j+\rho} = \frac{1}{\gamma}\sum_{i=1}^{\gamma} \mathrm{A}_{j+\rho,i}$, $j = 1,2,\dots$.	the function $\mathcal{HH}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$
Δ3.1.2	$\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E}_i,q_i,d_i)$ in(Δ1), $\forall\, i = 1,2,\dots,r$.	$\mathcal{E} = min\{\mathcal{E}_i\}, q = min\{q_i\}, d = min\{d_i\}$ and $1 \le i \le \gamma$.	the function $\mathcal{HH}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$in (Δ7)

Δ3.2. "$\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ is closed under convex linear combinations".

Theorem Δ3.2.1. Let it be the functions $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall\, i = 1,2$. Then the function $h\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ where $h\mathcal{H}_i(\varpi)$ is defined by

$$h\mathcal{H}_i(\varpi) = (1-\mathrm{v})\mathcal{H}_1(\varpi) + \mathrm{v}\mathcal{H}_2(\varpi),$$

$0 \le \mathrm{v} < 1$.

proof Δ3.2.1. First we calculate

$$h\mathcal{H}_i(\varpi) = \frac{1}{\varpi^\rho} + \sum_{j=1}^{\infty}\left[(1-\mathrm{v})\mathrm{A}_{j+\rho,1} + \mathrm{v}\mathrm{A}_{j+\rho,2}\right]\varpi^{j+\rho}$$

from Theorem Δ2.1, we get

$$\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\left[(1-\mathrm{v})\mathrm{A}_{j+\rho,1} + \mathrm{v}\mathrm{A}_{j+\rho,2}\right]$$
$$= (1-\mathrm{v})\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,1}$$
$$+\mathrm{v}\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\,\mathrm{A}_{j+\rho,2}$$
$$\le (1-\mathrm{v})\mathcal{E}(1+q)(\rho-d) + \mathrm{v}\mathcal{E}(1+q)(\rho-d)$$
$$\le \mathcal{E}(1+q)(\rho-d).$$

Hence $h\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$.

Theorem Δ3.2.2. Let it be the functions $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall i = 1,2,$. Then the function $g\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ where $g\mathcal{H}_i(\varpi)$ is defined by

$$g\mathcal{H}_i(\varpi) = \frac{1}{2}[\mathcal{H}_1(\varpi) + \mathcal{H}_2(\varpi)]$$

proof Δ3.2.2. First we calculate

$$g\mathcal{H}_i(\varpi) = \frac{1}{\varpi^\rho} + \frac{1}{2}\sum_{j=1}^{\infty}\left[\mathrm{A}_{j+\rho,1} + \mathrm{A}_{j+\rho,2}\right]\varpi^{j+\rho},$$

Let's

$$\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\frac{1}{2}\left[\mathrm{A}_{j+\rho,1} + \mathrm{A}_{j+\rho,2}\right]$$
$$= \frac{1}{2}\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,1}$$
$$+\frac{1}{2}\sum_{j=1}^{\infty}(j+\rho)(1+q_i\mathcal{E}_i)T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,2}$$
$$\le \frac{1}{2}\mathcal{E}(1+q)(\rho-d) + \frac{1}{2}\mathcal{E}(1+q)(\rho-d)$$
$$= \mathcal{E}(1+q)(\rho-d).$$

Hence $g\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$.

https://doi.org/10.61856/4cq33h32

Table 2: Comparison between Theorem Δ3.2.1 and Theorem Δ3.2.2
($\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E}, q, d)$ is closed under convex linear combinations)

theorem	let	where	then
Δ3.2.1	$\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall\, i = 1,2$.	$h\mathcal{H}_i(\varpi)$ is defined by $h\mathcal{H}_i(\varpi) = (1-v)\mathcal{H}_1(\varpi) + v\mathcal{H}_2(\varpi)$, $0 \le v < 1$.	the unction $h\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$
Δ3.2.2	$\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall\, i = 1,2$.	$g\mathcal{H}_i(\varpi)$ is defined by $g\mathcal{H}_i(\varpi) = \frac{1}{2}[\mathcal{H}_1(\varpi) + \mathcal{H}_2(\varpi)]$	the function $g\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$

Δ3.3. " convolution properties for $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ ".

Theorem Δ3.3.1. Let it be the functions $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall i = 1,2$. Then the function $(\mathcal{H}_1 * \mathcal{H}_2)(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},Q,d)$ in (Δ2) where

$$Q = \frac{(1+\rho)(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+q)^2(\rho-d)}{\mathcal{E}^2(1+q)^2(\rho-d) - (1+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

proof Δ3.3.1. we find the largest value of Q such that.

$$\sum_{j=1}^{\infty}(j+\rho)(1+Q\mathcal{E})T(\Theta,\mathcal{M},\rho)\left[\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}\right] \le \mathcal{E}(1+Q)(\rho-d).$$

From $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in(Δ1), $\forall i = 1,2$, and from Theorem Δ2.1, we get

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,i} \le \mathcal{E}(1+q)(\rho-d), \forall i = 1,2.$$

From the Cauchy- Schwarz inequality we find

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}} \le \mathcal{E}(1+q)(\rho-d). \qquad (\Delta 8)$$

So, we will show

$$\sum_{j=1}^{\infty}\frac{(j+\rho)(1+Q\mathcal{E})T(\Theta,\mathcal{M},\rho)}{\mathcal{E}(1+Q)(\rho-d)}\left[\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}\right] \le \sum_{j=1}^{\infty}\frac{(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}{\mathcal{E}(1+q)(\rho-d)}\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}}.$$

This would be, if

$$\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}} \le \frac{(1+Q)(1+q\mathcal{E})}{(1+q)(1+Q\mathcal{E})},$$

by (Δ8), we get

$$\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}} \le \frac{\mathcal{E}(1+q)(\rho-d)}{(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)},$$

therefore, if

$$\frac{\mathcal{E}(1+q)(\rho-d)}{(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)} \le \frac{(1+Q)(1+q\mathcal{E})}{(1+q)(1+Q\mathcal{E})}. \qquad (\Delta 9)$$

by (Δ9), we get

$$Q \le \frac{(j+\rho)(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+q)^2(\rho-d)}{\mathcal{E}^2(1+q)^2(\rho-d) - (j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

Function definition

$M_1(j)$

https://doi.org/10.61856/4cq33h32

$$= \frac{(j+\rho)(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+q)^2(\rho-d)}{\mathcal{E}^2(1+q)^2(\rho-d) - (j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)},$$

we notice that the function $M_1(j)$ is increasing.
For $j \geq 1$, hence

$$Q = \frac{(1+\rho)(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+q)^2(\rho-d)}{\mathcal{E}^2(1+q)^2(\rho-d) - (1+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

Theorem Δ3.3.2. Let it be the functions $\mathcal{H}_1(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ and $\mathcal{H}_2(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},Q,d)$ in $(\Delta 1)$. Then the function $(\mathcal{H}_1 * \mathcal{H}_2)(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},\emptyset,d)$ in $(\Delta 2)$ where

$$\emptyset = \frac{(1+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+Q)(1+q)(\rho-d)}{\mathcal{E}^2(1+Q)(1+q)(\rho-d) - (1+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

proof Δ3.3.2. we find the largest value of $\emptyset$ such that.

$$\sum_{j=1}^{\infty}(j+\rho)(1+\emptyset\mathcal{E})T(\Theta,\mathcal{M},\rho)\left[\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}\right] \leq \mathcal{E}(1+\emptyset)(\rho-d).$$

From Theorem Δ2.1 and $\mathcal{H}_1(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in $(\Delta 1)$, we get

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,1} \leq \mathcal{E}(1+q)(\rho-d)$$

and $\mathcal{H}_2(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},Q,d)$ in $(\Delta 1)$, we get

$$\sum_{j=1}^{\infty}(j+\rho)(1+Q\mathcal{E})T(\Theta,\mathcal{M},\rho)\mathrm{A}_{j+\rho,2} \leq \mathcal{E}(1+Q)(\rho-d).$$

From the Cauchy- Schwarz inequality we find

$$\sum_{j=1}^{\infty}(j+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}} \leq \mathcal{E}(1+q)(\rho-d). \qquad (\Delta 8)$$

So, we will show

$$\sum_{j=1}^{\infty}\frac{(j+\rho)(1+\emptyset\mathcal{E})T(\Theta,\mathcal{M},\rho)}{\mathcal{E}(1+\emptyset)(\rho-d)}\left[\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}\right]$$

$$\leq \sum_{j=1}^{\infty}\frac{(j+\rho)T(\Theta,\mathcal{M},\rho)}{\mathcal{E}(\rho-d)}\sqrt{\frac{(1+Q\mathcal{E})(1+q\mathcal{E})}{(1+Q)(1+q)}\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}}.$$

This would be, if

$$\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}} \leq \frac{(1+\emptyset)\sqrt{(1+Q\mathcal{E})(1+q\mathcal{E})}}{(1+\emptyset\mathcal{E})\sqrt{(1+Q)(1+q)}},$$

by $(\Delta 10)$, we get

$$\sqrt{\mathrm{A}_{j+\rho,1}.\mathrm{A}_{j+\rho,2}} \leq \frac{\mathcal{E}(\rho-d)\sqrt{(1+Q)(1+q)}}{(j+\rho)T(\Theta,\mathcal{M},\rho)\sqrt{(1+Q\mathcal{E})(1+q\mathcal{E})}},$$

therefore, if

$$\frac{\mathcal{E}(\rho-d)\sqrt{(1+Q)(1+q)}}{(j+\rho)T(\Theta,\mathcal{M},\rho)\sqrt{(1+Q\mathcal{E})(1+q\mathcal{E})}} \leq \frac{(1+\emptyset)\sqrt{(1+Q\mathcal{E})(1+q\mathcal{E})}}{(1+\emptyset\mathcal{E})\sqrt{(1+Q)(1+q)}}. \qquad (\Delta 11)$$

by $(\Delta 11)$, we get

$$\emptyset \leq \frac{(j+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+Q)(1+q)(\rho-d)}{\mathcal{E}^2(1+Q)(1+q)(\rho-d) - (j+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

Function definition

$$M_2(j) = \frac{(j+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+Q)(1+q)(\rho-d)}{\mathcal{E}^2(1+Q)(1+q)(\rho-d) - (j+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)},$$

we notice that the function $M_2(j)$ is increasing.
For $j \geq 1$, hence

$$\emptyset = \frac{(1+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+Q)(1+q)(\rho-d)}{\mathcal{E}^2(1+Q)(1+q)(\rho-d) - (1+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}.$$

Table 3: Comparison between Theorem Δ3.3.1 and Theorem Δ3.3.2
(convolution properties for $\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$)

theorem	let	where	then
Δ3.3.1	$\mathcal{H}_i(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ in $(\Delta 1)$, $\forall\, i = 1,2.$	$Q = \frac{(1+\rho)(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+q)^2(\rho-d)}{\mathcal{E}^2(1+q)^2(\rho-d) - (1+\rho)(1+q\mathcal{E})T(\Theta,\mathcal{M},\rho)}$	$(\mathcal{H}_1 * \mathcal{H}_2)(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},Q,d)$ in $(\Delta 2)$
Δ3.3.2	$\mathcal{H}_1(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ and $\mathcal{H}_2(\varpi)$	$\emptyset = \frac{(1+\rho)(1+Q\mathcal{E})(1+q\mathcal{E})^2 T(\Theta,\mathcal{M},\rho) - \mathcal{E}(1+Q)(1+q)(\rho-d)}{[illegible]}$	$(\mathcal{H}_1 * \mathcal{H}_2)(\varpi) \in$

Conclusion. In this research, the tributary factor was calculated on the multivalent meromorphic function, and thus we obtained a new function, and we introduced a new subclass and defined the class to which the new functions belong, and we found the necessary and sufficient condition for the functions to belong to the class, and we studied and obtained new results for the properties of these functions in this class, including $\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},\mathrm{q},\mathrm{d})$ is closed under arithmetic mean and $\Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},\mathrm{q},\mathrm{d})$ is closed under convex linear combinations and convolution properties for $\mathcal{H}_{\mathrm{i}}(\varpi) \in \Im\Gamma_{\mathcal{M}}^{\Theta}(\mathcal{E},q,d)$ and as explained further in the tables 1-2-3 respectively.

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